Topology Notes

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Contents

These notes are from my spring 2021 Introduction to Topology course. This course was taught by Dr. Gary Richardson at UCF out of C. Wayne Patty's *Foundations of Topology*, 2nd ed.

1 Preliminaries

Definition 1.0.1 (De Morgan's Laws)**.** De Morgan's Laws give us relations between unions, intersections, and complements of sets.

Let X be a set and $A_j \subset X, j \in J$ (J is an **index set**). Then,

1. $\left(\bigcup_{j\in J} A_j\right)^C = \bigcap_{j\in J} A_j^C$
2. $\left(\bigcap_{j\in J} A_j\right)^C = \bigcup_{j\in J} A_j^C$

Lemma. Let $f : x \to y,\ A_j \subseteq X,\ A \subseteq X,\ B_j \subseteq Y$. Then,

1. $f\left(\bigcup_{j\in J} A_j\right) = \bigcup_{j\in J} f(A_j)$
2. $f\left(\bigcap_{j\in J} A_j\right) = \bigcup_{j\in J} f(A_j)$
3. $f^{-1}\left(\bigcup_{j\in J} B_j\right) = \bigcup_{j\in J} f^{-1}(B_j)$
4. $f^{-1}\left(\bigcap_{j\in J} B_j\right) = \bigcap_{j\in J} f^{-1}(B_j)$
5. $f^{-1}\big(B^C\big) = \big(f^{-1}(B)\big)^C$
6. $f\big(A^C\big) = (f(A))^C$ no nice connection

2 Metrics and Norms

Definition 2.0.1 (Norm)**.** A norm "$\|\cdot\|$" is a map

$$\|\cdot\| \to [0, \infty]$$

Given a vector space V over $\mathbb{R}$; $x \in V$ and $\alpha \in \mathbb{R}$, a norm has the following properties:

1. $\|x\| = 0 \iff x = 0$
2. $\|\alpha x\| = |\alpha|\|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

Lemma. Let V be a vector space with norm $\|\cdot\|$. Define $d = \|x - y\|$. Then d is a metric on V.

Definition 2.0.2 (Metric)**.** Given a set X and $x \in X$, $d : x \times x \to (0, \infty)$ is a metric if:

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z$

Ex. 2.0.1 (Norm in $\mathbb{R}^n$)**.**

$$\|x \in \mathbb{R}^n\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

Ex. 2.0.2 (Euclidean Metric in $\mathbb{R}^n$)**.** $d(x, y) = \|x - y\| = \left(\sum\limits_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$

Ex. 2.0.3 (l_2 Metric)**.** Given

$$l_2(\mathbb{R}) = \left\{ (x_i) \middle| x_i \in \mathbb{R}; \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$$

Define $\|x\| = \left(\sum\limits_{i=1}^{\infty} x_i^2\right)^{1/2}$. Then the metric is

$$d(x, y) = \|x - y\| = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2\right)^{1/2}$$

.

Ex. 2.0.4 (l_1 "Taxi Cab" Metric)**.** Given

$$l_1(\mathbb{R}) = \left\{ (x_i) \middle| x_i \in \mathbb{R}; \sum_{i=1}^{\infty} |x_i| < \infty \right\}$$

Define $\|x\| = \sum\limits_{i=1}^{\infty} |x_i|$. Then the metric is

$$\rho(x, y) = \|x - y\| = \sum_{i=1}^{\infty} |x_i - y_i|$$

.

Ex. 2.0.5 (Metric on Continuous Functions)**.** On an interval $[a, b]$:
Given

$$\{f | f : [a, b] \to \mathbb{R}; f \text{ continuous}\}$$

Define $\|f\| = \int_a^b |f(x)| \mathrm{d}x$. Then the metric is

$$d(f, g) = \|f - g\| = \int_a^b |f(x) - g(x)| \mathrm{d}x$$

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Ex. 2.0.6 (Square Metric)**.**

$$\sigma(x, y) = \|x - y\| = \max \{|x_i - y_i| | i \in [1, n]\}$$

Ex. 2.0.7 (Discrete Metric)**.**

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} \tag{1}$$

Definition 2.0.3 (Open Ball)**.** Given a point x, and open ball around it is denoted by

$$B(x, \delta) = \{y \in X | d(x, y) < \delta; \delta > 0; x \in X\}$$

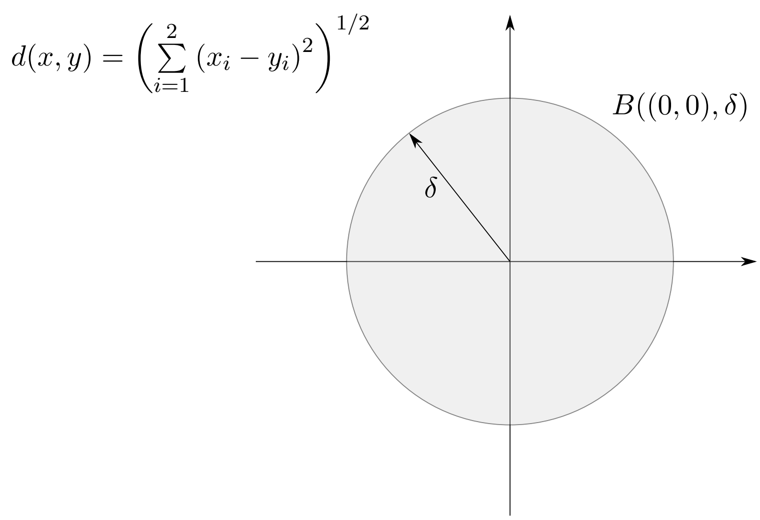

Figure 1: An open ball with the Euclidean Metric

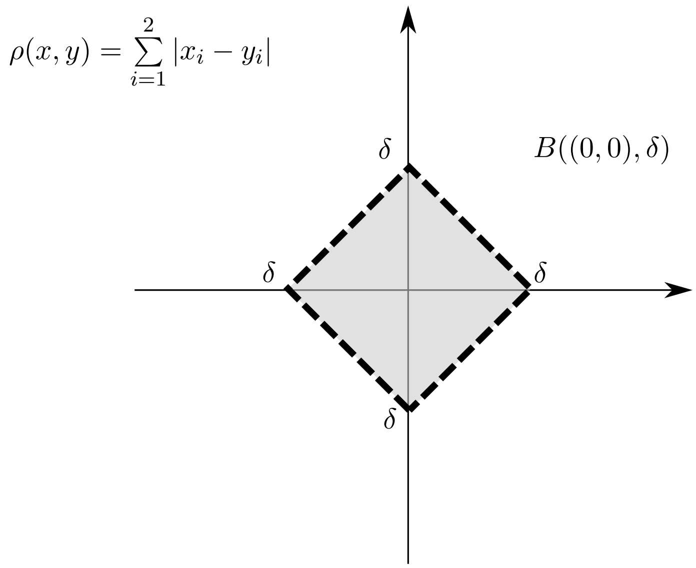

Figure 2: An open ball with the l_1 Metric

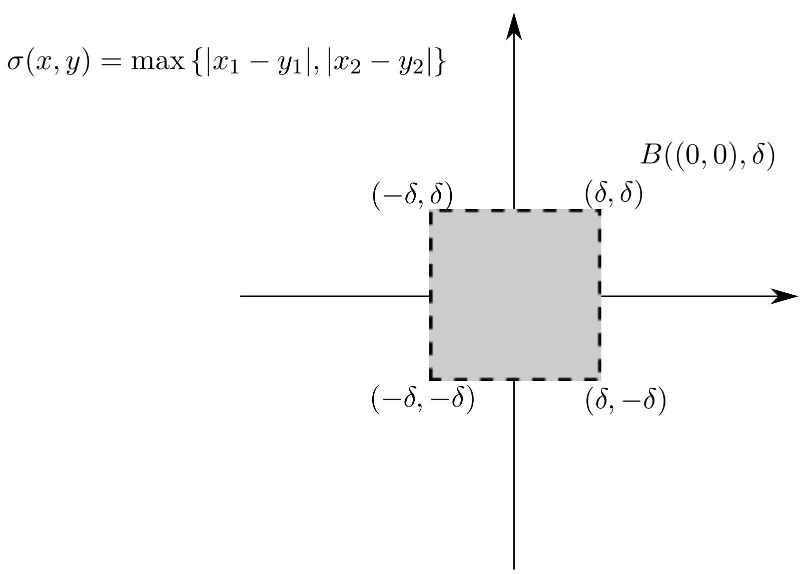

Figure 3: An open ball with the Square Metric