Advanced Calculus Notes

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January 17, 2021

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These notes are from my spring 2021 Advanced Calculus course. This course was taught by Dr. Sona Swanson at UCF out of Fitzpatrick's *Advanced Calculus*, 2nd ed.

1 Preliminaries

In order to rigorously develop analysis, one needs to understand the foundation on which it is constructed–$\mathbb{R}$. We will assume that the field of real numbers $\mathbb{R}$ satisfies three groups of axioms: the **Field Axioms**, the **Positivity Axioms**, and the **Completeness Axiom**.

The Field Axioms

1. Commutativity of Addition: $a + b = b + a$
2. Associativity of Addition: $(a + b) + c = a + (b + c)$
3. Additive Identity: $0 + x = x$
4. Additive Inverse: $x + -x = 0$
5. Commutativity of Multiplication: $ab = ba$
6. Associativity of Multiplication: $(ab)c = a(bc)$
7. Multiplicative Identity: $1 \cdot x = x$
8. Multiplicative Inverse: All a other than the Additive Inverse have a b s.t. $a \cdot b = 1$. Call this inverse a^{-1}.
9. Distributive Property: $a(b + c) = ab + ac$
10. The Nontrivial Assumption: $1 \neq 0$, i.e., the Multiplicative Identity $\neq$ Additive Identity

The Positivity Axioms

1. $a, b \in \mathbb{P} \implies ab \in \mathbb{P} \quad \text{and} \quad a + b \in \mathbb{P}$
2. For any $a \in \mathbb{R}$, exactly one of a or $-a \in \mathbb{P}$ unless $a = 0$

Properties of Inequalities

Using the Field and Positivity Axioms, one can establish the following familiar properties of inequalities.

1. For each $a \in \mathbb{R}, a \neq 0 \implies a^2 > 0$.
2. For each $a \in \mathbb{P}, a^{-1} > 0$
3. If $a > b$, then

$$\begin{aligned} ac > bc \quad &\text{if} \quad c > 0, \quad \text{and} \\ ac < bc \quad &\text{if} \quad c < 0 \end{aligned}$$

Ordering $\mathbb{R}$

Define $a > b \quad \text{if} \quad a + (-b) \in \mathbb{P} \quad \text{and} \quad a \geq b \quad \text{if} \quad a + (-b) \in \mathbb{P} \quad \text{or} \quad a + (-b) = 0$. Similarly define for $b > a \quad \text{and} \quad b \geq a$. Put into words, a is greater than b if the sum of a and $-b$ is positive. If the sum is zero, then they are equal.

2 Tools for Analysis

Definition 2.0.1 (Inductive Set)**.** A set S of real numbers is said to be **inductive** if

1. the number $1 \in S$
2. if the number $x \in S \quad \text{then} \quad x + 1 \in S$

Ex. 2.0.1. The set of real numbers $\mathbb{R}$ is inductive.

Definition 2.0.2 (The Natural Numbers)**.** The natural numbers $\mathbb{N}$ are the intersection of all the inductive subsets of $\mathbb{R}$

Remark. There is an interesting consequence that follows from Definition 2.0.2 : any inductive set of natural numbers is identically $\mathbb{N}$.

What about mathematical induction?

Definition 2.0.3 (Mathematical Induction)**.** Let $S(n)$ be a mathematical property or assertion, where $n \in \mathbb{N}$. Suppose $S(1)$ is true, and that if $S(k)$ is true then $S(k+1)$ is true. Then, $S(n)$ will be true $\forall n$.

Given $\mathbb{N}$, how do we construct $\mathbb{Z}$, the integers?

Definition 2.0.4 (The Integers)**.** $\mathbb{Z}$ is the natural numbers, their additive inverses, and 0. It is closed under addition and multiplication.

$$\mathbb{Z} = \{a, -a | a \in \mathbb{N}\} \cup \{0\}$$

We are now ready to construct the rationals $\mathbb{Q}$.

Definition 2.0.5 (The Rationals)**.**) $\mathbb{Q}$ is the set of numbers that can be created by the ratio of two integers.

$$\left\{ x \middle| x = m \cdot (n)^{-1}; m, n \in \mathbb{Z}; n \neq 0 \right\}$$